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# On the Ratio Vectors of Polynomials

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## 1. INTRODUCTION

Let $p(x)$ be a polynomial of degree $n \geq 2$ with $n$ distinct real roots $r_1 < r_2 < \cdots < r_n$. Let $x_1 < x_2 < \cdots < x_{n-1}$ be the critical points of $p$, and define the *ratios*

$$\sigma_i = \frac{x_i - r_i}{r_{i+1} - r_i}, \qquad i = 1, \ldots, n-1. \tag{1}$$

$(\sigma_1, \ldots, \sigma_{n-1})$ is called the *ratio vector* of $p$, and $\sigma_k$ is called the $k$th ratio. In [2] Andrews recently proved

$$\frac{1}{n-i+1} < \sigma_i < \frac{i}{i+1}. \tag{2}$$

Andrews also defined the subsets of $R^{n-1}$, $X_n$ and $Y_n$.

$$X_n = \prod_{i=1}^{n-1} \left( \frac{1}{n-i+1}, \frac{i}{i+1} \right),$$

and $Y_n$ is the set of elements in $R^{n-1}$ which are the ratio vectors of polynomials with $n$ distinct real zeroes. For $n = 3$ Andrews proved in [2] that

$$(1 - \sigma_1)\sigma_2 = \tfrac{1}{3}. \tag{3}$$

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By (2), $Y_n \subseteq X_n$, and in [2] the following question was posed: Does $Y_n = X_n$? Equation (3) actually shows immediately that $Y_3 \neq X_3$. We also show that $Y_4 \neq X_4$. Indeed, for $n = 3$, we show (Theorem 1) that $Y_3$ is *precisely* the subset of $X_3$ given by the curve in (3). For $n = 4$, we show (Theorem 2) that every point in $Y_4$ lies on the zero set of a polynomial $Q(x, y, z)$ of degree 9.

We also investigate the *monotonicity* of the ratios. For $n = 3$, it follows immediately from (2) that $\sigma_1 < \sigma_2$. For $n = 4$, we also prove (Theorem 3) that $\sigma_1 < \sigma_2 < \sigma_3$. The monotonicity fails in general for each $n \geq 5$ (see Theorem 6). However, if the roots of a polynomial are real, distinct, and *equispaced*, then the ratios are monotonic (see Theorem 4).

To prove the lack of monotonicity of the ratios for $n \geq 5$, we investigate the ratio vectors of polynomials with three real, distinct zeros: $p(x) = x^m(x - r)(x - s)$, where $0 < r < s$, and $m \geq 2$. Similar to Theorem 1, we give a complete characterization of the ratios $\sigma_1 = x_1/r$ and $\sigma_2 = (x_2 - r)/s - r$. More precisely, let $Y_m$ = set of elements in $R^2$ which are the ratio vectors of polynomials $p(x) = x^m(x - r)(x - s)$, with $0 < r < s$ and $m \geq 2$. We prove (Theorem 5) that $Y_m = Z_m$, where

$$Z_m = \left\{(\sigma_1, \sigma_2) \colon \frac{m}{m+2} < \sigma_1 < \frac{m}{m+1} \quad \text{and} \quad \sigma_2 = \frac{1}{(m+2)(1-\sigma_1)}\right\}.$$

It is probably impossible to completely characterize the ratios of $n$th degree polynomials with all real, distinct zeros. Indeed, even for $n = 4$, we have only shown that $Y_4 \subseteq X_4 \cap \{(x, y, z) \colon Q(x, y, z) = 0\}$. It would be nice to prove, for any $n$, that $(\sigma_1, \ldots, \sigma_{n-1})$ is a subset of the zero set of a polynomial $Q$ in $n - 1$ variables, $Q$ independent of $r_1, \ldots, r_n$.

## 2. RATIO VECTORS FOR CUBICS AND QUARTICS

Recall that $X_3 = \{(x, y) \in R^2 \colon \frac{1}{3} < x < \frac{1}{2}$ and $\frac{1}{2} < y < \frac{2}{3}\}$, and $Y_3 = \{(\sigma_1, \sigma_2) \colon (\sigma_1, \sigma_2)$ is the ratio vector of a cubic polynomial with all real, distinct zeros$\}$.

THEOREM 1. *Let* $Z_3 = \{(\sigma_1, \sigma_2) \in X_3 \colon (1 - \sigma_1)\sigma_2 = \frac{1}{3}\}$. *Then* $Y_3 = Z_3$.

*Proof.* If $p(x) = (x - r_1)(x - r_2)(x - r_3)$, then a simple computation shows that $p'(x) = 3(x - x_1)(x - x_2)$, where

$$x_1 = \frac{r_1 + r_2 + r_3}{3} - \frac{\sqrt{r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_1 r_3 - r_2 r_3}}{3}$$
$$x_2 = \frac{r_1 + r_2 + r_3}{3} + \frac{\sqrt{r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_1 r_3 - r_2 r_3}}{3}. \tag{4}$$

Suppose now that $(s, t) \in Z_3$, so that $\frac{1}{3} < s < \frac{1}{2}$ (it is also true that $\frac{1}{2} < t < \frac{2}{3}$, but we do not need to use that fact). Now fix any two values for $r_1$ and $r_2$. For convenience let $r_1 = 0$ and $r_2 = 1$. Then

$$x_1 = \frac{r_3 + 1}{3} - \frac{\sqrt{r_3^2 - r_3 + 1}}{3} \qquad \text{and}$$
$$x_2 = \frac{r_3 + 1}{3} + \frac{\sqrt{r_3^2 - r_3 + 1}}{3}, \qquad \text{and}$$

if $\sigma_1$ is the first ratio of $p$, then $\sigma_1 = x_1$. Now let $f(r_3) = (r_3 + 1)/3 - \sqrt{r_3^2 - r_3 + 1}/3$. Then $f(1) = \frac{1}{3}$, and $\lim_{r_3 \to \infty} f(r_3) = \frac{1}{2}$. Hence, by the Intermediate Value Theorem, $f(r_3) = s$ for some $r_3 > 1$. That is, the monic polynomial $p(x)$ with roots 0, 1, and $r_3$ has $s$ as its first ratio. Let $s = (x_2 - r_2)/(r_3 - r_2) = (x_2 - 1)/(r_3 - 1)$. By (3), with $n = 3$, $(1 - \sigma_1)\sigma_2 = \frac{1}{3}$. Since $t \in Z_3$, $(1 - s)t = \frac{1}{3}$. Hence $t = \sigma_2$, and that proves that $Z_3 \subseteq Y_3$. Equations (2) and (3) imply that $Y_3 \subseteq Z_3$, and hence $Y_3 = Z_3$. This proves that $Z_3 \subseteq Y_3$. Equations (2) and (3) imply that $Y_3 \subseteq Z_3$, and hence $Y_3 = Z_3$.

*Remark.* In the proof above, we could have fixed any two values of $r_1$ and $r_2$, and the proof would follow in a similar, but slightly more complicated, fashion.

Now we consider the quartic case where $p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$. Since $p(x)$ and $p(x + c)$ have the same ratio vectors for any constant $c$, we may assume, without loss of generality, that $r_2 = 0$. Then we have

$$p(x) = (x - r_1)x(x - r_3)(x - r_4), \text{ with } r_1 < 0 < r_3 < r_4. \tag{5}$$

Then

$$p'(x) = 4x^3 - 3x^2(r_1 + r_3 + r_4) + 2x(r_1 r_3 + r_1 r_4 + r_3 r_4) - r_1 r_3 r_4.$$

Letting $x_1 < x_2 < x_3$ denote the critical points of $p$, we also have

$$\begin{aligned} p'(x) &= 4(x - x_1)(x - x_2)(x - x_3) \\ &= 4x^3 - 4x^2(x_1 + x_2 + x_3) \\ &\quad + 4x(x_1 x_2 + x_1 x_3 + x_2 x_3) - 4x_1 x_2 x_3. \end{aligned}$$

Equating coefficients gives:

$$x_1 x_2 x_3 = \tfrac{1}{4} r_1 r_3 r_4 \tag{6}$$

$$x_1 + x_2 + x_3 = \tfrac{3}{4}(r_1 + r_3 + r_4) \tag{7}$$

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = \tfrac{1}{2}(r_1 r_3 + r_1 r_4 + r_3 r_4). \tag{8}$$

Now $x_1 = (1 - \sigma_1) r_1$, $x_2 = \sigma_2 r_3$, and $x_3 = (r_4 - r_3)\sigma_3 + r_3$. Substituting into the LHS of Eqs. (6)–(8) yields

$$\left((1 - \sigma_1)\sigma_2\sigma_3 - \tfrac{1}{4}\right) r_4 + (1 - \sigma_1)\sigma_2(1 - \sigma_3) r_3 = 0, \tag{9}$$

$$\left(\tfrac{1}{4} - \sigma_1\right) r_1 + \left(\tfrac{1}{4} + \sigma_2 - \sigma_3\right) r_3 + \left(\sigma_3 - \tfrac{3}{4}\right) r_4 = 0, \tag{10}$$

$$\begin{aligned} &(\sigma_1(\sigma_2 - \sigma_3 + 1) - \sigma_2 + \sigma_3 + 1) r_1 r_3 + (\sigma_1\sigma_3 - \sigma_3 + 2) r_1 r_4 \\ &\quad + \sigma_2(\sigma_3 - 1) r_3^2 + (2 - \sigma_2\sigma_3) r_3 r_4 = 0. \end{aligned} \tag{11}$$

One can easily solve (9) and (10) for $r_1$ and $r_3$ in terms of $r_4$. We used Derive to do this, giving

$$\begin{aligned} r_1 &= -r_4 \frac{\begin{array}{r} 16\sigma_1\sigma_2^2\sigma_3 - 24\sigma_1\sigma_2\sigma_3 + 12\sigma_1\sigma_2 - 16\sigma_2^2\sigma_3 + 24\sigma_2\sigma_3 \\ -8\sigma_2 - 4\sigma_3 + 1 \end{array}}{4\sigma_2(1 - \sigma_3)(\sigma_1 - 1)(4\sigma_1 - 1)}, \\ r_3 &= -r_4 \left( \frac{4\sigma_2\sigma_3(\sigma_1 - 1) + 1}{4\sigma_2(1 - \sigma_3)(\sigma_1 - 1)} \right). \end{aligned} \tag{13}$$

Now substitute (12) and (13) into (11), which gives an equation of the form $Q(\sigma_1, \sigma_2, \sigma_3) r_4^2 = 0$, which implies that $Q(\sigma_1, \sigma_2, \sigma_3) = 0$. This defines the smooth surface in $R^3$ on which the ratio vectors $(\sigma_1, \sigma_2, \sigma_3)$ must lie. We found $Q$ explicitly, using Derive again to do the algebraic simplifications. $Q$ is a polynomial of total degree 9 in $\sigma_1$, $\sigma_2$, and $\sigma_3$, and it has 38 nonzero terms. Thus we have proved

THEOREM 2. *$Y_4$ is a subset of the surface $Q = 0$, where $Q$ is the following polynomial*:

$$\begin{aligned} Q := (x, y, z) \to & -1 - 384x^2y^4z^2 - 288x^2y^3z \\ & + 704x^2y^3z^2 + 2x - 128y^4z^2 - 96y^3z - 264y^2z^2 \\ & + 144y^2z + 80yz^2 - 54yz + 8xz^2 - 18xy - 24x^2y^2 \\ & + 40xy^2 + 8x^2y + 116xyz - 128xyz^2 - 376xy^2z \\ & + 296x^2y^2z + 48x^2yz^2 + 568xy^2z^2 + 288xy^3z \\ & - 368x^2y^2z^2 - 832xy^3z^2 - 56x^2yz + 384xy^4z^2 \\ & - 64x^3y^2z + 96x^3y^3z + 64x^3y^2z^2 - 192x^3y^3z^2 \\ & + 128x^3y^4z^2 - 8z^2 - 10xz - 16y^2 + 320y^3z^2 + 8y \\ & + 6z \end{aligned}$$

We used Maple to factor $Q$, and it has two irreducible factors:

$$\begin{aligned} (1 - 4y + 4xy)\big(& 32x^2y^3z^2 + 24x^2y^2z - 48x^2y^2z^2 \\ & +16x^2yz^2 - 16x^2yz - 64xy^3z^2 - 48xy^2z \\ & +120xy^2z^2 - 64xyz^2 - 6xy + 52xyz + 2x - 10xz \\ & +8xz^2 + 32y^3z^2 + 24y^2z - 72y^2z^2 + 48yz^2 + 4y \\ & -30yz - 1 + 6z - 8z^2\big). \end{aligned}$$

*Remark*. Theorem 2 shows immediately that $Y_4 \neq X_4$ since the zero set of a polynomial must have measure 0 in $R^3$, while $X_4$ has positive measure.

EXAMPLE. Let $r_1 = -1$, $r_2 = 0$, $r_3 = 1$, and $r_4 = 2$, and hence $p(x) = x(x^2 - 1)(x - 2)$. The roots of $p'(x)$ are

$$x_1 = \frac{1}{2} - \frac{\sqrt{5}}{2},\ x_2 = \frac{1}{2},\ \text{and}\ x_3 = \frac{1}{2} + \frac{\sqrt{5}}{2},$$

which implies that

$$\sigma_1 = \frac{3}{2} - \frac{\sqrt{5}}{2},\ \sigma_2 = \frac{1}{2},\ \text{and}\ \sigma_3 = -\frac{1}{2} + \frac{\sqrt{5}}{2}.$$

One can check that substituting these values into $Q$ does give 0.

## 3. MONOTONICITY OF RATIOS

For $n = 3$, since $\sigma_1 < \frac{1}{2} < \sigma_2$, it is trivial that $\sigma_1 < \sigma_2$. This result also holds for $n = 4$, but the proof is not quite so trivial:

THEOREM 3. *Let $n = 4$, and let $p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$ be any polynomial with all real, distinct roots. Then the ratios are increasing—i.e., $\sigma_1 < \sigma_2 < \sigma_3$.*

*Proof.* Write $p'(x) = 4(x - x_1)(x - x_2)(x - x_3)$. As earlier, we have (without assuming $r_2 = 0$):

$$x_1 x_2 x_3 = \tfrac{1}{4}(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4). \tag{14}$$

Now assume, without loss of generality, that $r_2 = 0$. Using $x_1 = (1 - \sigma_1)r_1$, $x_2 = \sigma_2 r_3$, and $x_3 = (1 - \sigma_3)r_3 + \sigma_3 r_4$, we have $(1 - \sigma_1)r_1\sigma_2 r_3((1 - \sigma_3)r_3 + \sigma_3 r_4) = \frac{1}{4}r_1 r_3 r_4$, which implies that

$$(1 - \sigma_1)\sigma_2(1 - \sigma_3)r_3 + \left((1 - \sigma_1)\sigma_2\sigma_3 - \tfrac{1}{4}\right)r_4 = 0.$$

Since $0 < r_3$, $0 < r_4$, and $(1 - \sigma_1)\sigma_2(1 - \sigma_3) > 0$, $(1 - \sigma_1)\sigma_2\sigma_3 - \frac{1}{4} < 0$.

Also, since $(1 - \sigma_1)\sigma_2(1 - \sigma_3)r_3 = (\frac{1}{4} - (1 - \sigma_1)\sigma_2\sigma_3)r_4$ and $r_3 < r_4$, $\frac{1}{4} - (1 - \sigma_1)\sigma_2\sigma_3 < (1 - \sigma_1)\sigma_2(1 - \sigma_3)$, which implies that $\frac{1}{4} < (1 - \sigma_1)\sigma_2$. Hence

$$\begin{aligned}\frac{\sigma_2}{\sigma_1} &> \frac{1/4}{(1 - \sigma_1)\sigma_1} > \frac{1}{4}\min_{x \in R}\left(\frac{1}{x(1 - x)}\right)\\ &= \frac{1}{4}\frac{1}{\max\limits_{x \in R}(x(1 - x))} = 1.\end{aligned}$$

To prove that $\sigma_3/\sigma_2 > 1$, assume that $r_3 = 0$ in (14). The proof then follows in exactly the same fashion.

The following example shows that Theorem 2 does *not* hold for $n = 5$.

*Remark.* Note that by (2), $\sigma_1 < \frac{1}{2} < \sigma_{n-1}$ for any $n$. So the ratios cannot be nonincreasing, and to prove they are nonmonotonic it suffices to prove that they are not nondecreasing either.

EXAMPLE. Let $p(x) = x(x - .1)(x - .2)(x - 2)(x - 3)$. By numerically approximating the critical points of $p$, one can show that $\sigma_3 \approx .66947$ and $\sigma_4 \approx .63688$, so that $\sigma_3 > \sigma_4$.

To show in general that the ratios are not monotonic for any $n \geq 5$, one considers the polynomial $p(x) = x^m(x - r)(x - s)$. For suitable $r$ and $s$, $p$ will have nonmonotonic ratios. A small perturbation of $p$ then gives a polynomial with distinct real roots and nonmonotonic ratios. We do this in the next section. It is true, however, that for each $n$ there are polynomials $p$ with all real, distinct roots, such that the ratios of $p$ are monotonic. Indeed, we prove

THEOREM 4. *Suppose that the roots of the polynomial $p(x)$ are all real, distinct, and equispaced. Then the ratios of $p$ are monotonic.*

*Proof.* Let $r_1 < r_2 < \cdots < r_n$ be the roots of $p$, and let $d = r_{k+1} - r_k$, $k = 1, \dots, n-1$. Then $\sigma_k = (x_k - r_k)/d$ for any $k$, where the $x_k$ are the critical points of $p$. By [3, E.12, p. 26]

$$\min_{1 \le k \le n-2} (x_{k+1} - x_k) > \min_{1 \le i \le n-1} (r_{i+1} - r_i).$$

Hence,

$$x_{k+1} - x_k > d = r_{k+1} - r_k,$$

which implies that

$$x_{k+1} - r_{k+1} > x_k - r_k,$$

and hence $\sigma_{k+1} > \sigma_k$.

We investigate briefly the ratio vectors of $p(x) = x^m(x-r)(x-s)$ in the next section.

## 4. RATIO VECTORS OF POLYNOMIALS WITH THREE DISTINCT REAL ROOTS AND A ZERO OF MULTIPLICITY $m$

THEOREM 5. *Let $p(x) = x^m(x-r)(x-s)$, with $0 < r < s$ and $m \ge 2$. Write $p'(x) = (m+2)x^{m-1}(x-x_1)(x-x_2)$, and define the ratios $\sigma_1 = x_1/r$ and $\sigma_2 = (x_2 - r)/(s-r)$. Let $Y_m$ =* set *of elements in $R^2$ which are the ratio vectors of polynomials $p(x) = x^m(x-r)(x-s)$, with $0 < r < s$ and $m \ge 2$. Then $Y_m = Z_m$, where*

$$Z_m = \left\{ (\sigma_1, \sigma_2) \colon \frac{m}{m+2} < \sigma_1 < \frac{m}{m+1} \text{ and } \sigma_2 = \frac{1}{(m+2)(1-\sigma_1)} \right\}.$$

*Remark.* If $m = 1$, we obtain Andrew's bounds for $\sigma_1$ (see (2)).

However, for $m \ge 2$, $m/(m+2)$ is a better lower bound than taking limits of polynomials with distinct real roots and using (2) (in that case one gets $\frac{1}{3} \le \sigma_1$).

*Proof.* The proof that $m/(m+2) < \sigma_1 < m/(m+1)$ is similar to the proof of (2) in [2]. Let $s \to r$ (for fixed $r$) in a decreasing manner. Then $p(x) \to x^m(x-r)^2$, and hence $p'(x) \to x^{m-1}(x-r)(2x+m(x-r))$.

By the root-dragging theorem (see [1]), $x_1$ *decreases* to the root of $2x+m(x-r)$, which is $mr/(m+2)$. Hence $\sigma_1$ decreases to $m/(m+2)$, and that must be the lower bound for $\sigma_1$. In a similar fashion, one can prove that $\sigma_1 < m/(m+1)$. Or, one could take limits of polynomials with distinct real roots, the first $m$ of which are close to 0, and use (2) (note that in that case $\sigma_1$ is really $\sigma_m$). However, that would only give $\sigma_1 \le m/(m+1)$. We also have the relations $((m+1)/(m+2))(r+s) = x_1 + x_2$ and $(m/(m+2))rs = x_1 x_2$. This implies that

$$\left(\frac{m+1}{m+2}\right)(r+s) = \sigma_1 r + (s-r)\sigma_2 + r$$

and

$$\left(\frac{m}{m+2}\right)rs = \sigma_1 r((s-r)\sigma_2 + r).$$

Simplifying yields

$$(a - \sigma_1 + \sigma_2 - 1)r + (a - \sigma_2)s = 0$$
$$(\sigma_1\sigma_2 - \sigma_1)r + (b - \sigma_1\sigma_2)s = 0,$$

where $a = (m+1)/(m+2)$ and $b = m/(m+2)$. To get nontrivial solutions to this homogeneous linear system in $r$ and $s$, we set the determinant equal to 0. This gives

$$\sigma_2 = \frac{a(b+\sigma_1) - b(1+\sigma_1)}{2a\sigma_1 - b - \sigma_1^2} = \frac{1}{(m+2)(1-\sigma_1)}.$$

That proves that $Y_m \subseteq Z_m$.

Now assume, without loss of generality, that $r = 1$, so that $p(x) = x^m(x-1)(x-s)$. Then $\sigma_1 = x_1$, and a simple computation gives

$$x_1 = \frac{(m+1)(s+1) - \sqrt{m^2(s-1)^2 + 2m(s-1)^2 + (s+1)^2}}{2(m+2)} := f(s)$$

Now $f(1) = m/(m+2)$ and $\lim_{s\to\infty} f(s) = m/(m+1)$. Hence, by the Intermediate Value Theorem, if $m/(m+2) < t < m/(m+1)$, then $\exists s > 1$ such that $f(s) = \sigma_1$. Hence $t$ is the first ratio of $p$, and by what we have just proved, if $\sigma_2 = (x_2 - r)/(s-r) = (x_2 - 1)/(s-1)$ is the other ratio of $p$, then $\sigma_2 = 1/(m+2)(1-t)$. That proves that $Z_m \subseteq Y_m$ and completes the proof of Theorem 5.

We can now prove that the ratios of polynomials with distinct real roots are not monotonic in general for any $n \geq 5$.

THEOREM 6. *Let $n \geq 5$. Then there exists a polynomial $p$ of degree $n$, with distinct real roots, such that $\sigma_{n-2} > \sigma_{n-1}$.*

*Proof.* First consider $p(x) = x^m(x-r)(x-s)$, with $0 < r < s$ and $m = n - 2 \geq 3$. Let $\sigma_1 = x_1/r$, $\sigma_2 = (x_2 - r)/(s - r)$, and $g(x) = 1/(m+2)x$. By Theorem 5, $(\sigma_1, \sigma_2)$ lies on the graph of $y = g(x)$. The line $y = x$ intersects the graph of $g$ at

$$x = \frac{m + 2 \pm \sqrt{m^2 - 4}}{2(m+2)} = \frac{1}{2} \pm \frac{\sqrt{m^2 - 4}}{2(m+2)}.$$

In addition, the line $y = x$ lies above the graph of $g$ between

$$u_1 = \frac{1}{2} - \frac{\sqrt{m^2 - 4}}{2(m+2)} \text{ and } u_2 = \frac{1}{2} + \frac{\sqrt{m^2 - 4}}{2(m+2)}.$$

Now a simple computation shows that

$$\frac{1}{2} - \frac{\sqrt{m^2 - 4}}{2(m+2)} < \frac{m}{m+2} < \frac{1}{2} + \frac{\sqrt{m^2 - 4}}{2(m+2)}$$

whenever $m > 2$. Hence there exists $\sigma_1$ such that $u_1 < \sigma_1 < u_2$ and $m/(m+2) < \sigma_1 < m/(m+1)$. By Theorem 5, $\sigma_1$ is the first ratio of some polynomial $p(x) = x^m(x-r)(x-s)$, with $0 < r < s$, so that $\sigma_1 = x_1/r$. Then the second ratio $\sigma_2 = (x_2 - r)/(s - r)$ must satisfy $\sigma_2 = 1/(m+2)(1 - \sigma_1) = g(\sigma_1)$, again by Theorem 5. Since $y = x$ lies above the graph of $g$ for $u_1 < x < u_2$, $\sigma_2 < \sigma_1$. Now let $q(x)$ be a polynomial of degree $n$ with $n$ distinct real roots $r_1 < \cdots < r_{n-3} < 0 < r < s$. If the $r_k$ are chosen sufficiently close to 0, then $\sigma_{n-2} > \sigma_{n-1}$, and hence the ratios of $q$ are not monotonic.

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